The winding number.

Monday, December 7, 2020 9:06 AM

Consider Cauchy integral of $\varphi(z) \equiv 1$.

Lemma. Let γ be a (piecewise smooth) closed curve. Then $\forall z_0 \notin \gamma$,

$$\frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-z_0} \in \mathbb{Z}.$$

Notation $n(\gamma, z_0)$ - index of the point z_0 with respect to curve γ, $n(\gamma,z_0) := \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-z_0}$. $(z_0 \notin \gamma)$

For closed curves: also called winding number of γ around z_0.

If γ is closed, by Lemma, $n(\gamma, z_0) \in \mathbb{Z}$. $(z_0 \notin \gamma)$.

Non-proof of Lemma. $\log(z-a)' = \frac{1}{z-a}$, ~~$z_0$~~, if $z: [a,b] \to \gamma$ - a parameterization;

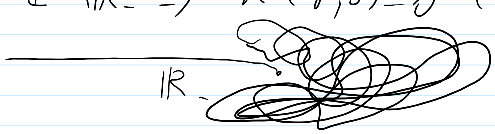
then $\oint_\gamma \frac{dz}{z-z_0} = \log(z(b)-z_0) - \log(z(a)-z_0) \in 2\pi i \mathbb{Z}$. $(z(b)=z(a))$.

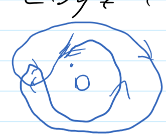
Can be made rigorous using coverings.

Bonus (rigorous!) If $\gamma \subset \Omega$, $\exists\, \ell: \Omega \to \mathbb{C}$ - a function with $e^{\ell(z)} = z - z_0$, (a branch of $\log(z-z_0)$).

$$\oint_\gamma \frac{dz}{z-z_0} = \ell(z(b)-z_0) - \ell(z(a)-z_0) = 0$$

Example. $\gamma \subset \mathbb{C} \setminus \mathbb{R}_- \Rightarrow n(\gamma, 0) = 0$ ($\ell(z) = \operatorname{Log} z$ is a branch





Proof of Lemma.

Let $z(t)$ be a parametrization of γ, $z: [a,b] \to \gamma$

$$h(t) := \int_a^t \frac{z'(s)}{z(s)-z_0}\,ds \qquad h(b) = 2\pi i\, n(\gamma, z_0).$$

Then $\left(\frac{e^{h(t)}}{z(t)-z_0}\right)' = e^{h(t)}\frac{z'(t)}{(z(t)-z_0)^2} - e^{h(t)}\frac{z'(t)}{(z(t)-z_0)^2} = 0.$

so $\frac{e^{h(t)}}{z(t)-z_0} = \frac{e^{h(a)}}{z(a)-z_0} = \frac{1}{z(a)-z_0}$. In particular:

$e^{h(b)} = \frac{1}{}$ $\quad z(b) = z(a)$, so $e^{h(b)} = 1 \Leftrightarrow h(b) \in 2\pi i \mathbb{Z}$

$z(a)-z_0$ $\qquad$ $z(a)-z_0$

$$\frac{e^{h(b)}}{z(b)-z_0} = \frac{1}{z(a)-z_0} \qquad z(b)=z(a), \text{ so } e^{h(b)}=1 \Leftrightarrow h(b)\in 2\pi i\,\mathbb{Z}$$

<u>Properties</u>. 1) If D is a connected component of $\mathbb{C}\setminus\gamma$, then $\forall z_1, z_2 \in D$: $n(\gamma, z_1) = n(\gamma, z_2)$

<u>Proof</u>. $z \to n(\gamma, z)$ - analytic function (Cauchy integral), $\in \mathbb{Z}$.

D - connected $\Rightarrow$ $n(\gamma, z) \equiv$ const $\blacksquare$

2) $n(\gamma, z) = -n(-\gamma, z)$, $\quad n(\gamma_1+\gamma_2, z) = n(\gamma_1, z) + n(\gamma_2, z)$

<u>Proof</u>. Integrate $\blacksquare$

3) If z is in unbounded component of $\mathbb{C}\setminus\gamma$, then <u>($\gamma$ - closed)</u>

$n(\gamma, z) = 0$

<u>Proof</u>. Note: for $|z| > 2\max_{w\in\gamma} |w|$,

$$\left|\frac{1}{z-w}\right| < \frac{2}{|z|}, \ \forall w\in\gamma. \text{ So } \ n(\gamma, z) = \left|\frac{1}{2\pi i}\oint_\gamma \frac{dw}{w-z}\right| \le$$

$$\frac{\ell(\gamma)}{2\pi}\cdot\frac{2}{|z|} = \frac{2\,\ell(\gamma)}{\pi|z|} \to 0 \text{ as } |z|\to\infty. \text{ But } n(\gamma, z) \equiv \text{const in}$$

the unbounded component, so $n(\gamma, z) \equiv 0$ $\blacksquare$

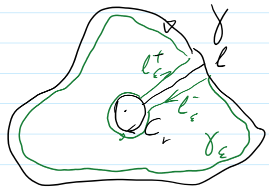
<u>Observation</u>: Let γ be a counter-clockwise oriented circle $\{z_0 + re^{it}, 0 \le t \le 2\pi\}$.

Then for $|z-z_0| > r$, $\quad n(\gamma, z) = 0$

$$|z-z_0| < r, \quad n(\gamma, z) = n(\gamma, z_0) = \frac{1}{2\pi i}\oint_{C_r} \frac{d\xi}{\xi - z_0} = 1.$$

The same is true for any Jordan closed curve γ.

Cutting construction:



For z in the interior of γ, let $r < \operatorname{dist}(z,\gamma)$, C_r - circle of radius r, oriented clockwise

ℓ - the shortest interval from C_r to γ.

Curve γ_ε for small $\varepsilon > 0$ defined on the picture.

z is in unbounded component of γ_ε.

So $n(\gamma_\varepsilon, z) = 0$.

z is in unbounded component of $\mathbb{C} \setminus \gamma_\varepsilon$.

Let $\varepsilon \to 0$. $n(\ell_\varepsilon^+, z) = \int_{\ell_\varepsilon^+} \frac{dw}{w-z} = \int_\ell \frac{dw}{w+i\varepsilon-z} \xrightarrow[\varepsilon\to 0]{} \int_\ell \frac{dw}{w-z} = n(\ell, z)$

—— '' $n(\ell_\varepsilon^-, z) \to -n(\ell, z)$.

So $n(\gamma_\varepsilon, z) \to n(\gamma, z) + n(C_r, z) = n(\gamma, z) - 1 \Rightarrow$
$n(\gamma, z) = 1$ $\blacksquare$